A NEW CONSTRUCTION FOR CYCLOIDS

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IN the following lines is shown a new way for generating cycloids, and a simple method of constructing such curves. The advantage is looked for in the fact that, here, with the points of the curve are given simultaneously the respective tangents; and this is a constructional simplification. The kinematical aspect of the problem is also of interest.

Beginning with the simplest case,* consider a circle rolling uniformly over a straight line, and simultaneously a point P describing a simple harmonic motion (abbreviated SHM) along a diameter of the rolling circle. If to a complete revolution of the circle corresponds a complete period of the SHM, and if both motions begin at the same instant, and start from the same point, then the equations in rectangular coordinates of the path of P assume the form

$$x = r(a - \sin a \cos a),$$
$$y = r(1 - \cos^2 a),$$

where use is made of the relation that exists between SHM and uniform circular motion, so that P is found by dropping a perpendicular from the contact of the circle with the fixed line to the diameter in which the SHM takes place.

These equations may be written in the following way:

$$x = \frac{r}{2}(2a - \sin 2a),$$

$$y = \frac{r}{2}(1 - \cos 2a).$$

Putting $\frac{r}{2} = r'$, $2a = a'$, we finally get

$$x = r'(a' - \sin a'),$$
$$y = r'(1 - \cos a'),$$

* A proof for the construction in the case of the common cycloid was published by the author in *The American Mathematical Monthly*, February, 1909.

which is the common parameter form of the equations of the cycloid. We thus see that the cycloid may be defined as the path of a point having a *SHM* along the diameter of a uniformly rolling circle. The cycloid thus generated is the same as if described by a point on the rim of a rolling circle of radius $r/2$.

For the tangent we get

$$\frac{dy}{dx} = \frac{\sin 2a}{1 - \cos 2a} = \cot a = \tan\left(\frac{\pi}{2} - a\right),$$

which says that the diameter in which the *SHM* occurs is always tangent to the curve; the points of the curve as well as their corresponding tangents are thus found at the same time. This fact is also evident from kinematical considerations.

As the next case we consider the rolling of a circle of radius r on a fixed circle of radius R, and at the same time a *SHM* along a diameter of the moving circle, the conditions imposed being the same as in the case of the cycloid. Denoting by a the angle formed by the line of centers of the circles with its original direction, we get for a point P of the resulting path

$$x = (R + r)\cos a - r\cos\frac{R}{r}\,a\,\cos\frac{R+r}{r}\,a\,,$$

$$y = (R + r)\sin a - r\cos\frac{R}{r}\,a\,\sin\frac{R+r}{r}\,a\,.$$

After some reductions these equations simplify to the following:

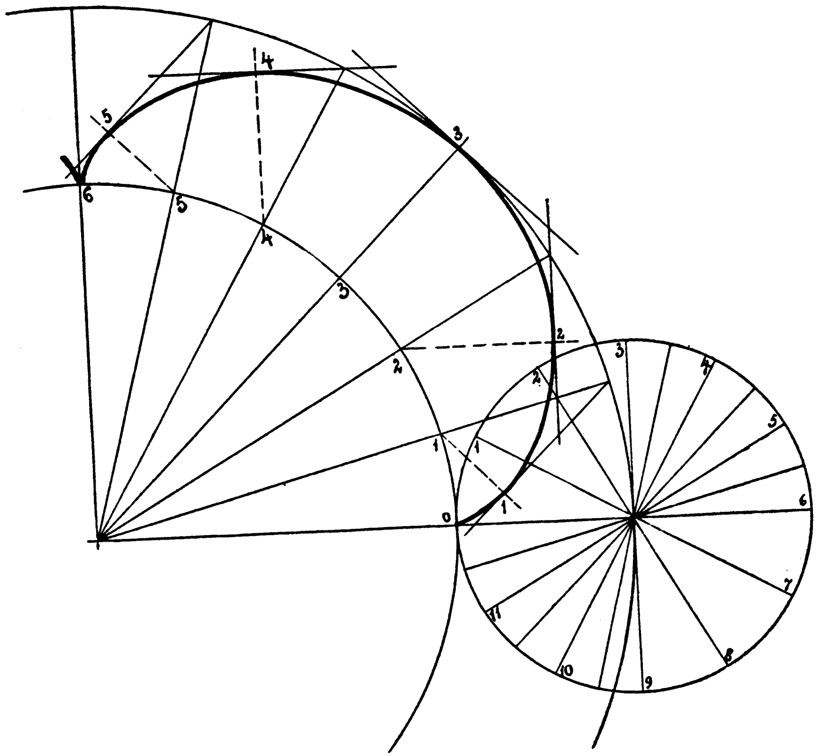
$$x = (R + r')\cos a - r'\cos\frac{R+r'}{r'}\,a\,,$$

$$y = (R + r')\sin a - r'\sin\frac{R+r'}{r'}\,a\,,$$

where

$$r' = \frac{r}{2}\,.$$

We thus see that the resulting path of a point describing a *SHM* under the stated conditions along the diameter of a circle of radius r rolling on another circle of radius R is an epicycloid. The epicycloid thus generated is



the same as if described by a point on the rim of a circle of radius $r/2$ rolling over one of radius R.

For the slope of the tangent we get

$$\frac{dy}{dx} = \tan \frac{R + r}{r} a = \tan \theta ,$$

where θ is the inclination of the diameter considered to Ox, and therefore the diameter in which the *SHM* takes place is tangent to the epicycloid for every

one of its points, so that we find simultaneously the points and the tangents to the curve.

In a similar manner we get for the case of a circle rolling on the inside of another circle, the *SHM* taking place along a diameter of the rolling circle under the same conditions as before,

$$x = (R - r)\cos a + r \cos \frac{R}{r} a \cos \frac{R - r}{r} a \,,$$

$$y = (R - r)\sin a - r \cos \frac{R}{r} a \sin \frac{R - r}{r} a,$$

and by reducing we get

$$x = (R - r')\cos a + r'\cos \frac{R - r'}{r'} a \,,$$

$$y = (R - r')\sin a - r'\sin \frac{R - r'}{r'} a,$$

where $r' = r/2$.

These equations express that the resulting path is an hypocycloid — the same as if described by a point on the rim of a circle of radius $r/2$ rolling over the inside of one of radius R.

For the tangent we get

$$\frac{dy}{dx} = - \tan \frac{R - r}{r} a = \tan \theta \,,$$

and therefore here, too, the diameter in which the *SHM* occurs is tangent to the hypocycloid for every one of its points.

The construction is similar to that in the case of the epicycloid.

It is also to be noticed that the points of the curve are here found *directly* as the intersection of the tangent and normal.

From the foregoing the conclusion seems to be justified that it is interesting and advantageous to define the cycloids as the resultant of a *SHM* combined with that of the rolling of a circle.

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